A discrete form of the Beckman-Quarles theorem for rational eight-space

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Abstract. Let $\mathbf{Q}$ be the field of rationals numbers. We prove that: (1) if $x, y \in \mathbf{R}^n$ $(n > 1)$ and $|x-y|$ is constructible by means of ruler and compass then there exists a finite set $S_{xy} \subseteq \mathbf{R}^n$ containing x and y such that each map from S_{xy} to $\mathbf{R}^n$ preserving unit distance preserves the distance between x and y, (2) if $x, y \in \mathbf{Q}^8$ then there exists a finite set $S_{xy} \subseteq \mathbf{Q}^8$ containing x and y such that each map from S_{xy} to $\mathbf{R}^8$ preserving unit distance preserves the distance between x and y.

Theorem 1 may be viewed as a discrete form of the classical Beckman-Quarles theorem, which states that any map from $\mathbf{R}^n$ to $\mathbf{R}^n$ $(2 \le n < \infty)$ preserving unit distance is an isometry, see [1]-[3]. Theorem 1 was announced in [9] and prove there in the case where $n = 2$. A stronger version of Theorem 1 can be found in [10], but we need the elementary proof of Theorem 1 as an introduction to Theorem 2.

Theorem 1. If $x, y \in \mathbf{R}^n$ $(n > 1)$ and $|x-y|$ is constructible by means of ruler and compass then there exists a finite set $S_{xy} \subseteq \mathbf{R}^n$ containing x and y such that each map from S_{xy} to $\mathbf{R}^n$ preserving unit distance preserves the distance between x and y.

Proof. Let us denote by D_n the set of all non-negative numbers d with the following property:

if $x, y \in \mathbf{R}^n$ and $|x-y| = d$ then there exists a finite set $S_{xy} \subseteq \mathbf{R}^n$ such that $x, y \in S_{xy}$ and any map $f : S_{xy} \to \mathbf{R}^n$ that preserves unit distance preserves also the distance between x and y.

Obviously $0, 1 \in D_n$. We first prove that if $d \in D_n$ then $\sqrt{2+2/n} \cdot d \in D_n$. Assume that $d > 0$, $x, y \in \mathbf{R}^n$ and $|x-y| = \sqrt{2+2/n} \cdot d$. Using the notation of Figure 1 we show that

$$S_{xy} := \bigcup \{S_{ab} : a, b \in \{x, y, \widetilde{y}, p_1, p_2, ..., p_n, \widetilde{p}_1, \widetilde{p}_2, ..., \widetilde{p}_n\}, |a-b| = d\}$$

satisfies the condition of Theorem 1. Figure 1 shows the case $n = 2$, but equations below Figure 1 describe the general case $n \ge 2$; z denotes the centre of the $(n-1)$-dimensional regular simplex $p_1 p_2 ... p_n$.

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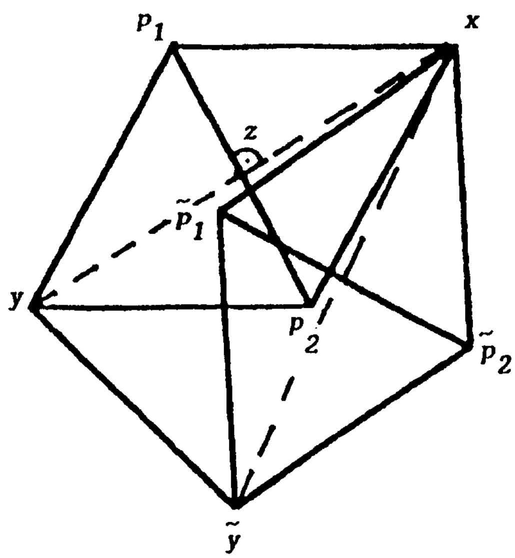

Figure 1

$$1 \le i < j \le n$$
$$|y - \widetilde{y}| = d,\ |x - p_i| = |y - p_i| = |p_i - p_j| = d = |x - \widetilde{p}_i| = |\widetilde{y} - \widetilde{p}_i| = |\widetilde{p}_i - \widetilde{p}_j|$$
$$|x - \widetilde{y}| = |x - y| = 2 \cdot |x - z| = 2 \cdot \sqrt{\tfrac{n+1}{2n}} \cdot d = \sqrt{2 + 2/n} \cdot d$$

Assume that $f : S_{xy} \to \mathbf{R}^n$ preserves distance 1. Since

$$S_{xy} \supseteq S_{y\widetilde{y}} \cup \bigcup_{i=1}^{n} S_{xp_i} \cup \bigcup_{i=1}^{n} S_{yp_i} \cup \bigcup_{1 \le i < j \le n} S_{p_i p_j}$$

we conclude that f preserves the distances between y and $\widetilde{y}$, x and p_i $(1 \le i \le n)$, y and p_i $(1 \le i \le n)$, and all distances between p_i and p_j $(1 \le i < j \le n)$. Hence $|f(y) - f(\widetilde{y})| = d$ and $|f(x) - f(y)|$ equals either 0 or $\sqrt{2 + 2/n} \cdot d$. Analogously we have that $|f(x) - f(\widetilde{y})|$ equals either 0 or $\sqrt{2 + 2/n} \cdot d$. Thus $f(x) \neq f(y)$, so $|f(x) - f(y)| = \sqrt{2 + 2/n} \cdot d$ which completes the proof that $\sqrt{2 + 2/n} \cdot d \in D_n$.

Therefore, if $d \in D_n$ then $(2 + 2/n) \cdot d = \sqrt{2 + 2/n} \cdot (\sqrt{2 + 2/n} \cdot d) \in D_n$.

We next prove that if $x, y \in \mathbf{R}^n$, $d \in D_n$ and $|x - y| = (2/n) \cdot d$ then there exists a finite set $Z_{xy} \subseteq \mathbf{R}^n$ containing x and y such that any map $f : Z_{xy} \to \mathbf{R}^n$ that preserves unit distance satisfies $|f(x) - f(y)| \le |x - y|$; this result is adapted from [3]. It is obvious in the case where $n = 2$, therefore we assume that $n > 2$ and $d > 0$. In Figure 2, z denotes the centre of the $(n-1)$-dimensional regular simplex $p_1 p_2 ... p_n$. Figure 2 shows the case $n = 3$, but equations below Figure 2 describe the general case where $n \ge 3$.

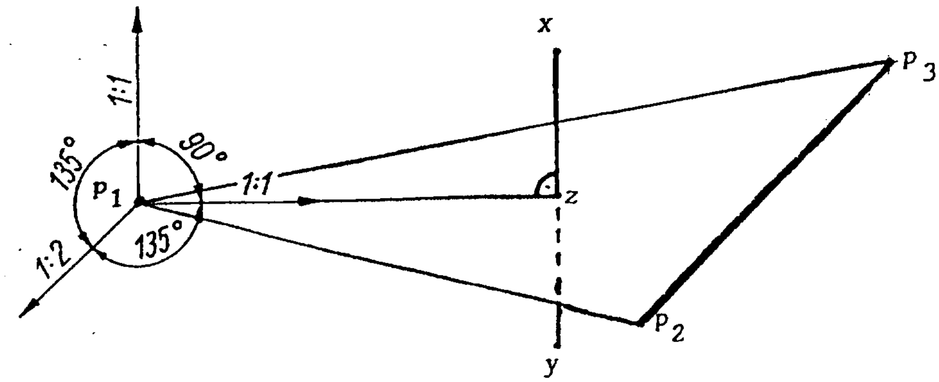

Figure 2

$$1 \leq i < j \leq n$$

$$|x - p_i| = |y - p_i| = d, \quad |p_i - p_j| = \sqrt{2 + 2/n} \cdot d, \quad |z - p_i| = \sqrt{1 - 1/n^2} \cdot d$$

$$|x - y| = 2 \cdot |x - z| = 2 \cdot \sqrt{|x - p_i|^2 - |z - p_i|^2} = 2 \cdot \sqrt{d^2 - (1 - 1/n^2) \cdot d^2} = (2/n) \cdot d$$

Define:

$$Z_{xy} := \bigcup_{1 \leq i < j \leq n} S_{p_i p_j} \cup \bigcup_{i=1}^{n} S_{x p_i} \cup \bigcup_{i=1}^{n} S_{y p_i}$$

If $f : Z_{xy} \to \mathbf{R}^n$ preserves distance 1 then $|f(x) - f(y)| = |x - y| = (2/n) \cdot d$ or $|f(x) - f(y)| = 0$, hence $|f(x) - f(y)| \leq |x - y|$.

If $d \in D_n$, then $2 \cdot d \in D_n$ (see Figure 3).

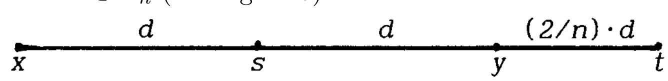

Figure 3

$$|x - y| = 2 \cdot d$$

$$S_{xy} = S_{xs} \cup S_{sy} \cup Z_{yt} \cup S_{xt}$$

From Figure 4 it is clear that if $d \in D_n$ then all distances $k \cdot d$ (where k is a positive integer) belong to D_n.

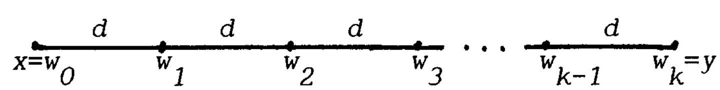

Figure 4

$$|x - y| = k \cdot d$$

$$S_{xy} = \bigcup \{S_{ab} : a, b \in \{w_0, w_1, ..., w_k\}, |a - b| = d \vee |a - b| = 2 \cdot d\}$$

From Figure 5 it is clear that if $d \in D_n$ then all distances d/k (where k is a positive integer) belong to D_n. Hence $\mathbf{Q} \cap (0, \infty) \subseteq D_n$.

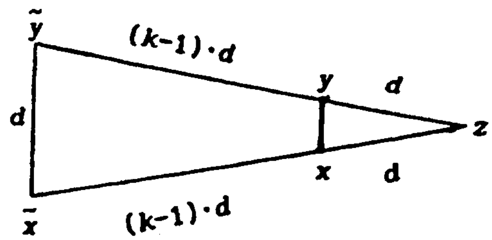

Figure 5
$|x-y| = d/k$
$S_{xy} = S_{\tilde{x}\tilde{y}} \cup S_{\tilde{x}x} \cup S_{xz} \cup S_{\tilde{x}z} \cup S_{\tilde{y}y} \cup S_{yz} \cup S_{\tilde{y}z}$

Observation. If $x, y \in \mathbf{R}^n$ $(n > 1)$ and $\varepsilon > 0$ then there exists a finite set $T_{xy}(\varepsilon) \subseteq \mathbf{R}^n$ containing x and y such that for each map $f : T_{xy}(\varepsilon) \to \mathbf{R}^n$ preserving unit distance we have $||f(x) - f(y)| - |x-y|| \le \varepsilon$.
Proof. It follows from Figure 6.

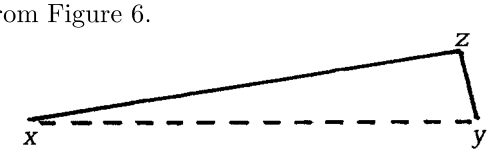

Figure 6
$|x-z|, |z-y| \in \mathbf{Q} \cap (0, \infty), \quad |z-y| \le \varepsilon/2$
$T_{xy}(\varepsilon) = S_{xz} \cup S_{zy}$

Note. The above part of the proof can be found in [10].

If $a, b \in D_n$, $a > b > 0$ then $\sqrt{a^2 - b^2} \in D_n$ (see Figure 7, cf.[9]).

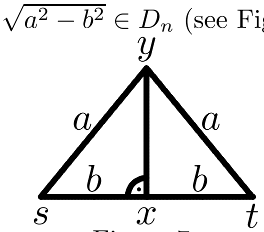

Figure 7
$|x-y| = \sqrt{a^2 - b^2}$
$S_{xy} = S_{sx} \cup S_{xt} \cup S_{st} \cup S_{sy} \cup S_{ty}$

Hence $\sqrt{3} \cdot a = \sqrt{(2 \cdot a)^2 - a^2} \in D_n$ and $\sqrt{2} \cdot a = \sqrt{(\sqrt{3} \cdot a)^2 - a^2} \in D_n$. Therefore $\sqrt{a^2 + b^2} = \sqrt{(\sqrt{2} \cdot a)^2 - (\sqrt{a^2 - b^2})^2} \in D_n$.

In Figure 8, z denotes the centre of the $(n-1)$-dimensional regular simplex

$p_1p_2...p_n$, $n = 2$, but equations below Figure 8 describe the general case where $n \geq 2$. This construction shows that if $a, b \in D_n$, $a > b > 0$, $n \geq 2$ then $a - b \in D_n$, hence $a + b = 2 \cdot a - (a - b) \in D_n$.

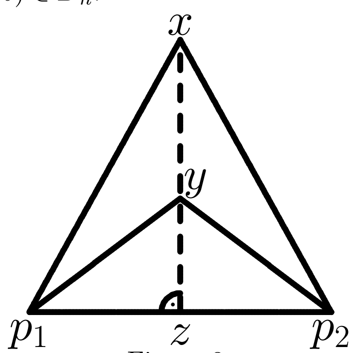

Figure 8

$$|x - y| = a - b, \quad |x - z| = a \in D_n, \quad |y - z| = b \in D_n$$
$$|p_i - p_j| = \sqrt{2 + 2/n} \in D_n, \quad |z - p_i| = \sqrt{1^2 - (1/n)^2} \in D_n, \quad 1 \leq i < j \leq n$$
$$|x - p_1| = \sqrt{|x - z|^2 + |z - p_1|^2} = ... = |x - p_n| = \sqrt{|x - z|^2 + |z - p_n|^2} \in D_n$$
$$|y - p_1| = \sqrt{|y - z|^2 + |z - p_1|^2} = ... = |y - p_n| = \sqrt{|y - z|^2 + |z - p_n|^2} \in D_n$$
$$S_{xy} = \bigcup_{1 \leq i < j \leq n} S_{p_ip_j} \cup \bigcup_{i=1}^{n} S_{xp_i} \cup \bigcup_{i=1}^{n} S_{yp_i} \cup T_{xy}(b)$$

In order to prove that $D_n \setminus \{0\}$ is a multiplicative group it remains to observe that if positive $a, b, c \in D_n$, then $\frac{a \cdot b}{c} \in D_n$ (see Figure 9, cf.[9]).

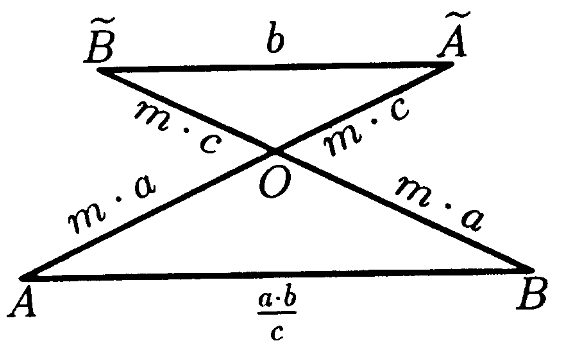

Figure 9

m is a positive integer

$b < 2 \cdot m \cdot c$

$$S_{AB} = S_{OA} \cup S_{OB} \cup S_{O\widetilde{A}} \cup S_{O\widetilde{B}} \cup S_{A\widetilde{A}} \cup S_{B\widetilde{B}} \cup S_{\widetilde{A}\widetilde{B}}$$

If $a \in D_n$, $a > 1$, then $\sqrt{a} = \frac{1}{2} \cdot \sqrt{(a+1)^2 - (a-1)^2} \in D_n$; if $a \in D_n$, $0 < a < 1$, then $\sqrt{a} = 1/\sqrt{\frac{1}{a}} \in D_n$. Thus D_n contains all non-negative real numbers contained in the real quadratic closure of $\mathbf{Q}$. This completes the proof.

Remark 1. Let $\mathbf{F} \subseteq \mathbf{R}$ is a euclidean field, i.e. $\forall x \in \mathbf{F} \exists y \in \mathbf{F}$ $(x = y^2 \vee x = -y^2)$ (cf. [6]). Our proof of Theorem 1 gives that if $x, y \in \mathbf{F}^n$ $(n > 1)$ and $|x - y|$ is constructible by means of ruler and compass then there exists a finite set $S_{xy} \subseteq \mathbf{F}^n$ containing x and y such that each map from S_{xy} to $\mathbf{R}^n$ preserving unit distance preserves the distance between x and y.

Theorem 2. If $x, y \in \mathbf{Q}^8$ then there exists a finite set $S_{xy} \subseteq \mathbf{Q}^8$ containing x and y such that each map from S_{xy} to $\mathbf{R}^8$ preserving unit distance preserves the distance between x and y.

Proof. Denote by R_8 the set of all $d \geq 0$ with the following property:

if $x, y \in \mathbf{Q}^8$ and $|x - y| = d$ then there exists a finite set $S_{xy} \subseteq \mathbf{Q}^8$ such that $x, y \in S_{xy}$ and any map $f : S_{xy} \to \mathbf{R}^8$ that preserves unit distance preserves also the distance between x and y.

Obviously $0, 1 \in R_8$. We need to prove that if $x \neq y \in \mathbf{Q}^8$ then $|x - y| \in R_8$. We show that configurations from Figures 1-5 and 7 (see the proof of Theorem 1) exist in $\mathbf{Q}^8$. We start from simple lemmas.

Lemma 1 (see [11]). If A and B are two different points of $\mathbf{Q}^n$ then the reflection of $\mathbf{Q}^n$ with respect to the hyperplane which is the perpendicular bisector of the segment AB, is a rational transformation (that is, takes rational points to rational points).

Lemma 2 (in the real case cf.[2] p.173 and [10]). If $A, B, \widetilde{A}, \widetilde{B} \in \mathbf{Q}^8$ and $|AB| = |\widetilde{A}\widetilde{B}|$ then there exists an isometry $I : \mathbf{Q}^8 \to \mathbf{Q}^8$ satisfying $I(A) = \widetilde{A}$ and $I(B) = \widetilde{B}$.
Proof. If $A = \widetilde{A}$ and $B = \widetilde{B}$ then $I = id(\mathbf{Q}^8)$. If $A = \widetilde{A}$ and $B \neq \widetilde{B}$ then the reflection of $\mathbf{Q}^8$ with respect to the hyperplane which is the perpendicular bisector of the segment $B\widetilde{B}$, satisfies the condition of Lemma 2 in virtue of Lemma 1. Assume that $A \neq \widetilde{A}$. Let $I_1 : \mathbf{Q}^8 \to \mathbf{Q}^8$ denote the reflection of $\mathbf{Q}^8$ with respect to the hyperplane which is the perpendicular bisector of the segment $A\widetilde{A}$. If $I_1(B) = \widetilde{B}$ then the proof is complete. In the opposite case let $B_1 = I_1(B)$, $B_1 \in \mathbf{Q}^8$ according to Lemma 1. Let $I_2 : \mathbf{Q}^8 \to \mathbf{Q}^8$ denote the reflection of $\mathbf{Q}^8$ with respect to the hyperplane which is the perpendicular bisector of the segment $B_1\widetilde{B}$. Since $|\widetilde{A}B_1| = |I_1(A)I_1(B)| = |AB| = |\widetilde{A}\widetilde{B}|$ we conclude that $I_2(\widetilde{A}) = \widetilde{A}$. Therefore $I = I_2 \circ I_1$ satisfies the condition of Lemma 2.

Corollary. Lemma 2 ensures that if some configuration from Figures 1-5 and 7 exists in $\mathbf{Q}^8$ for a fixed $x, y \in \mathbf{Q}^8$, then this configuration exists for any $x, y \in \mathbf{Q}^8$ with the same $|x - y|$.

Lemma 3. LAGRANGE'S FOUR SQUARE THEOREM. Every non-negative integer is the sum of four squares of integers, and therefore every non-negative

rational is the sum of four squares of rationals, see [8].

Lemma 4. If a, b are positive rationals and $b < 2a$ then there exists a triangle in $\mathbf{Q}^8$ with sides b, a, a.

Proof. Let $a^2 - (b/2)^2 = k^2 + l^2 + m^2 + n^2$ where k, l, m, n are rational according to Lemma 3. Then the triangle

$$[-b/2, 0, 0, 0, 0, 0, 0, 0]\ [b/2, 0, 0, 0, 0, 0, 0, 0]\ [0, k, l, m, n, 0, 0, 0]$$

has sides b, a, a.

Now we turn to the main part of the proof. Rational coordinates of the following configuration are taken from [11].

$x = [0, 0, 0, 0, 0, 0, 0, 0]$

$y = (3/8) \cdot [-3, 0, 0, 0, 1, 1, 1, -2]$ $\qquad \widetilde{y} = (1/6) \cdot [-8, 1, 1, 3, 1, 0, -1, -2]$

$p_1 = [-1, 0, 0, 0, 0, 0, 0, 0]$ $\qquad p_2 = (1/2) \cdot [-1, 1, 0, 0, 0, 0, 1, -1]$

$p_3 = (1/2) \cdot [-1, -1, 0, 0, 0, 0, 1, -1]$ $\qquad p_4 = (1/2) \cdot [-1, 0, 1, 0, 0, 1, 0, -1]$

$p_5 = (1/2) \cdot [-1, 0, -1, 0, 0, 1, 0, -1]$ $\qquad p_6 = (1/2) \cdot [-1, 0, 0, 1, 1, 0, 0, -1]$

$p_7 = (1/2) \cdot [-1, 0, 0, -1, 1, 0, 0, -1]$ $\qquad p_8 = (1/2) \cdot [-1, 0, 0, 0, 1, 1, 1, 0]$

Let $I : \mathbf{Q}^8 \to \mathbf{Q}^8$ denote the reflection with respect to the hyperplane which is the perpendicular bisector of the segment $y\widetilde{y}$. By Lemma 1 we have $\widetilde{p_i} = I(p_i) \in \mathbf{Q}^8$ $(1 \le i \le 8)$. It is easy to check that points $x, y, \widetilde{y}, p_i, \widetilde{p_i}$ $(1 \le i \le 8)$ form the configuration from Figure 1 for $d = 1$. The Corollary ensures that $3/2 = \sqrt{2 + 2/8} \cdot d = |x - y| \in R_8$.

Points $(3/2)x, (3/2)y, (3/2)\widetilde{y}, (3/2)p_i, (3/2)\widetilde{p_i}$ $(1 \le i \le 8)$ form the configuration from Figure 1 for $d = \sqrt{2 + 2/8} = 3/2$. The Corollary ensures that $2 + 1/4 = \sqrt{2 + 2/8} \cdot d = |(3/2) \cdot x - (3/2) \cdot y| \in R_8$.

The following points:

$p_1 = [-3/2, 0, 0, 0, 0, 0, 0, 0]$
$p_2 = [-3/4, 3/4, 0, 0, 0, 0, 3/4, -3/4]$
$p_3 = [-3/4, -3/4, 0, 0, 0, 0, 3/4, -3/4]$
$p_4 = [-3/4, 0, 3/4, 0, 0, 3/4, 0, -3/4]$
$p_5 = [-3/4, 0, -3/4, 0, 0, 3/4, 0, -3/4]$

$p_6 = [-3/4, 0, 0, 3/4, 3/4, 0, 0, -3/4]$
$p_7 = [-3/4, 0, 0, -3/4, 3/4, 0, 0, -3/4]$
$p_8 = [-3/4, 0, 0, 0, 3/4, 3/4, 3/4, 0]$
$x = [-3/4, 0, 0, 0, 1/4, 1/4, 1/4, -1/2]$
$y = [-15/16, 0, 0, 0, 5/16, 5/16, 5/16, -5/8]$
form the configuration from Figure 2 for $d = 1$. Therefore, in virtue of Corollary if $x, y \in \mathbf{Q}^8$ and $|x-y| = (2/8) \cdot d = 1/4$, then there exists a finite set $Z_{xy} \subseteq \mathbf{Q}^8$ containing x and y such that any map $f : Z_{xy} \to \mathbf{R}^8$ that preserves unit distance satisfies $|f(x) - f(y)| \leq |x - y|$.

As in the proof of Theorem 1 we can prove that $2 \in R_8$ and all integer distances belong to R_8. In the same way using the Corollary we can prove that all rational distances belong to R_8, because by Lemma 4 there exists a triangle in $\mathbf{Q}^8$ with sides $d, k \cdot d, k \cdot d$ (d, k are positive integers, see Figure 5).

Finally, we prove that $|x-y| \in R_8$ for arbitrary $x \neq y \in \mathbf{Q}^8$. It is obvious if $|x-y| = 1/2$ because $1/2$ is rational. Let us assume that $|x-y| \neq 1/2$. We have: $|x-y|^2 \in \mathbf{Q} \cap (0, \infty)$. Let $|x-y|^2 = k^2 + l^2 + m^2 + n^2$ where k, l, m, n are rationals according to Lemma 3.

The following points:

$s = [-||x-y|^2 - 1/4|, 0, 0, 0, 0, 0, 0, 0]$, $x = [0, 0, 0, 0, 0, 0, 0, 0]$

$t = [||x-y|^2 - 1/4|, 0, 0, 0, 0, 0, 0, 0]$, $y = [0, k, l, m, n, 0, 0, 0]$

form the configuration from Figure 7 for $a = ||x-y|^2 + 1/4| \in \mathbf{Q} \cap (0, \infty) \subseteq R_8$ and $b = ||x-y|^2 - 1/4| \in \mathbf{Q} \cap (0, \infty) \subseteq R_8$. The Corollary ensures that $|x-y| = \sqrt{a^2 - b^2} \in R_8$. This completes the proof of Theorem 2.

Remark 2. Theorem 2 implies that any map $f : \mathbf{Q}^8 \to \mathbf{Q}^8$ which preserves unit distance is an isometry.
Remark 3. It is known that the injection of $\mathbf{Q}^n$ ($n \geq 5$) which preserves the distances d and $d/2$ (d is positive and rational) is an isometry, see [5]. The general result from [7] implies that any map $f : \mathbf{Q}^n \to \mathbf{Q}^n$ ($n \geq 5$) which preserves the distances 1 and 4 is an isometry. On the other hand, from [4] (for $n = 1, 2$) and [5] (for $n = 3, 4$) it may be concluded that there exist bijections of $\mathbf{Q}^n$ ($n = 1, 2, 3, 4$) which preserve all distances belonging to $\{k/2 : k = 1, 2, 3, ...\}$ and which are not isometries.
Remark 4. J. Zaks informed (private communication, May 2000) the author that he proved the following:
1. (cf. Remark 3): Let k be any integer, $k \geq 2$; every mapping from $\mathbf{Q}^n$ to $\mathbf{Q}^n$,

$n \geq 5$, which preserves the distances 1 and k - is an isometry.
2. Theorem 2 holds for all even n of the form $n = 4t(t+1)$, $t \geq 2$, as well as for all odd values of n which are a perfect square greater than 1, $n = x^2$, and which, in addition, are of the form $n = 2y^2 - 1$. The construction is a modified version of the proof of Theorem 2.

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